**EXISTENCE OF SOLUTIONS FOR A CLASS OF NONLOCAL AND NON-HOMOGENEOUS EQUATIONS IN ORLICZ-SOBOLEV SPACE**

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***Abstract****. In this paper, we investigate the existence of multiple solutions for a class of non-homogeneous Kirchhoff type problems in Orlicz-Sobolev spaces. Our results are established by using the mountain pass theorem combined with the Ekeland variational principle.*

***Keywords:*** *Non-homogeneous operator; Orlicz-Sobolev spaces; Kirchhoff type problems; Variational methods*

1. **INTRODUCTION**

Let  be an open bounded subset of , with smooth boundary ,  is the outer unit normal derivative. Assume that is such that the mapping defined by



satisfies the condition : for all ,  is an odd, strictly increasing homeomorphism from onto .

In this work, we deal with the following Kirchhoff type problems with Neumann boundary condition where is a Carathéodory function, is a perturbation term and is a nondecreasing continuous function, and the functional *L* defined by

(1)

where

Problem [(1)](#page1) is a generalization of a model introduced by Kirchhoff [[16],](#page12) who studied the following equation

(3)



Problem (3) extends the classical D’Alembert’s wave equation by considering the effects of the changes in the length of the strings during the vibrations. Latter, the study of Kirchhoff type equations has already been extended to the case involving the *p-*Laplacian see [7, 13]. On the other hand, there is a great number of papers which have dealt with nonlocal p(x)-Laplacian equations, we refer the reader to [[3, 8, 18]](#page12) and the references therein for an overview on this subject.

We point out the fact that if , problem [(1)](#page1) becomes a nonlinear and non-homogeneous problem, which has been received considerable attention in recent years and studied by some authors in Orlicz-Sobolev spaces, see [[1, 4, 5, 23]](#page12) for the advances and references of this area. However, to our knowledge, there is not a great number of papers which have dealt with nonlocal and non-homogeneous equations through Orlicz-Sobolev spaces, we quoted some interesting papers [[6, 12, 14].](#page12) In [12], Figueiredo et al. studied the existence of solutions for a class of nonlocal and non-homogeneous equations using Krasnoselskiis genus. In [[14],](#page12) the authors considered problem [(1)](#page1) in the special case when In [[6],](#page12) the author studied the existence of solutions for the problem using a variational principle due to Ricceri [[21].](#page12) Motivated by the contributions cited above, in this paper we study the existence of nontrivial solution for the nonlocal problem [(1)](#page1) with perturbation *g* in Orlicz-Sobolev spaces. Our proofs are essentially based on the mountain pass theorem combined with the Ekeland variational principle.

1. **THE FUNCTIONAL RAMEWORK**

Here, we state some interesting properties of the theory of Orlicz- Sobolev spaces that will be useful to discuss problem [(1)](#page1). To be more precise, for the function  which satisfies condition H, we assume that the function

,

belongs to class (see [[20],](#page12) p. 33), i.e., the function satisfies the following conditions:

for all is a nondecreasing continuous function, with and whenever ,

 for every , is a measurable function.

Since  satisfies condition , we deduce that  is convex and increasing from .

Now, for the function  introduced above, we define the generalized Orlicz space

The space  is a Banach space endowed with the Luxemburg norm

or the equivalent norm (the Orlicz norm)

where  denotes the conjugate Young function of , that is, for each and

Furthermore, for and  conjugate Young functions, Holder’s inequality holds true

where C is a positive constant.

In this paper, we assume that there exist two positive constants  and  such that



The above relation implies that  satisfies the i.e

 (5)

where K is a positive constant.

Furthermore, we assume that  satisfies the following condition:

For each  the function  is convex on . (6)

Relations [(5)](#page3) and [(6)](#page3) assure that  is an uniformly convex space and thus, a reflexive space.

Here, we give some relations between the norm  and the modular:

.

**Proposition 2.1** [([19]).](#page12)*Assume that* [*(4)*](#page3)*, then the following relations hold:*

**

*for all  with  *

*for all  with *

We denote by  the corresponding generalized Orlicz-Sobolev space for problem (1), defined by

equipped with the equivalent norms



More precisely, (see, e.g [[19])](#page12) for every we have:

The generalized Orlicz-Sobolev space endowed with one of the above norms is a reflexive Banach space. In the following, we will use the norm on

**Proposition 2.2** [([19](#page12)]). *The following relations hold:*

**

*for all  with *

**

*for all  with *

**Remark 2.3**. Assuming that  and  belong to class  and there exists two positives constants k1; k2 and  a.e.  such that for all

(10)

then there exists a continuous embedding (see [[20, Theorem 8.5])](#page12). We point out that if [(10)](#page4) holds with then  is continuously embedded in 

In this paper, we study the problem [(1)](#page1) in the particular case when  satisfies:

(11)



where ** is a positive constant and the function  with Here,

We define the variable exponent Lebesgue space by This space endowed with the Luxemburg norm,



is a separable and reflexive Banach space. Denoting by  the conjugate space of  where ; for any  and  we have the following Holder type inequality

.

(12)

Now, we introduce the modular of the Lebesgue-Sobolev space  as mapping , defined by



In the following proposition, we give some relations between the Luxemburg norm and the modular.

**Proposition 2.4** [([10])](#page12)***.*** *If , then following properties hold true:*

1. **
2. **
3. **
4. **

Next, we define the variable exponent Sobolev space  by



endowed with the norm



The space  is separable and reflexive.

**Proposition 2.5** [([10])](#page12). *For such that  for all , there is a continuous (compact) embedding*

**

*where*

**

Before stating our results, we make the following assumptions on the functions and as follows:

(*m1*) is a nondecreasing continuous function, and there exists m0 > 0 such that  for all .

(*m2*) There exists such that



where  

uniformly for .

 uniformly for , where  such that 

 there exists  such that

and where  are given in (4) and assumption respectively.



We denote by *J* the energy functional associated with problem [(1)](#page1), that is,



where are defined as follows

,

(13)

where *L* defined by (2). Then, and is a weak solution of [(1)](#page1) if and only if u is a critical point of *J*. Moreover, we have

for all

We need the following lemma for the proofs of our main results.

**Lemma 2.6***. If the condition (m1) holds, then we have the following assertions:*

1. *I is sequentially weakly lower semicontinuous and coercive;*
2. * is strictly monotone;*

(iii) *I’ is of type (S+), i.e. if weakly in X, and*

**

*then*

Proof. (i) Since ,  is an increasing function on . By using the fact that the the functional *L* defined by [(2)](#page1) is sequentially weakly lower semicontinuous (see [[19]),](#page12) we see that I is sequentially weakly lower semicontinuous. Obviously, thanks to Proposition [2.2](#page4) and *(m1)*, for each  such that  we have

So, I is coercive.

(ii) Consider the functional L, whose  derivative at point  is given by

 for all .

Taking into account [[15,](#page12) Lemma 3.2],  is strictly monotone. So, by [[24,](#page12) Proposition 25.10], *L* is strictly convex. Moreover, since *M* is nondecreasing, is convex in . Thus, for every  with  and every  with , one has

From this, *I* is strictly convex, and, as already said, that  is strictly monotone.

(iii)From (ii), if as in and , we obtain

we also have

which yields

Since in *X*, it follows that is bounded sequence of real number. From the equivalent norms in relation [(9),](#page3) we see that and are bounded sequences of real numbers. Then, Proposition [2.1](#page3) yields that the sequence is bounded, up to subsequence, there is such that . The fact that *M* is continuous,

This and [(16)](#page6) imply

In the same way,

Then, we obtain by using relations [(17)](#page6) and (18) that

Using [[17,](#page12) Theorem 4] we obtain the strong convergence of in *X*, which ends the proof of (iii).

1. **MAIN RESULTS AND PROOFS**

Throughout the sequel and for simplicity, we use *ci* (*i* = 1,2, ..), to denote the general nonnegative or positive constants. The first result of this paper can be described as follows.

**Theorem 3.1.** *Assume that (m1), (f1), (f2) hold and suppose that . Then, problem* [*(1.1)*](#page1) *has a weak solution, provided that and .*

*Proof.* By conditions *(f1)* and *(f2)*, it follows that for any  there exists  depending on  such that

(20)

for all

Together with , and using Holder’s inequality (12), we have

Hence, for  sufficiently small, it follows that

By relation [(11)](#page4) and Remark 2.3 with , we deduce that the space *X* is continuously embedded in . On the other hand, Proposition [2.5](#page5) ensures that  is compactly embedded in . Thus, is compact. Then, there exist a positive constant *c2* such that

for all (22)

Then, by Propositions [2.2](#page4) and [2.4](#page4) the following hold

(23)

as since . By Lemma [2.6](#page6) (i), it is easy to verify that *J* is weakly lower semicontiguous. So *J* has a minimum point (see [[22,](#page12) Theorem 1.2]), which is a weak solution of problem [(1)](#page1).

Using the mountain pass theorem and Ekeland’s variational principle, we obtain the second main result.

**Theorem 3.2**. *Assume that (m1), (m2), (f1) - (f4) hold and suppose that and . Then there exists a constant such that problem* [*(1)*](#page1) *admits at least two nontrivial different solutions satisfying provided that .*

We first prove the following auxiliary lemmas which will be used in the proof of Theorem 3.2.

**Lemma 3.3**. *Under the conditions (m1), (f1) and (f2), there exist  such that for any ,  and for all with .*

Proof. From relation (23), the following hold

Since (we also have ), there exists such that

=

where

.

Then, taking we obtain that for and for all with .

**Lemma 3.4.** *Assume that conditions (m2) and (f3) hold. Then, there exists a nonnegative function  with such that , where is given in Lemma 3.3.*

*Proof.* Let



We have



for all  by . Hence,  for all , that is,

(24)

and

Now, we show that

, (25)

for all and

Indeed, from (4) for be fixed, we have

and it follows that relation (25) holds true. By the same way, integrating (m2) we obtain

, for all . (26)

Now, let a function  and . For and , in view of relations (24), (25) and (26), we obtain

since we deduce that as . So Lemma 3.4 is proved by choosing with large enough such that.

**Definition 3.5.** We say that *J* satisfises the Palais-Smale condition at level (briefly (PS)c ) on *X*, if any sequence , such that and as , possesses a convergent subsequence.

**Lemma 3.6.** *Assume that conditions (m1), (m2) and hold. Then the functional J satisfies the  condition with*

*Proof.* Consider a sequence which satisfies

, as (28)

Let us show that is bounded in *X*. Assume for convenience, according to , , , [(4)](#page3) and Proposition [2.2,](#page4) for large enough, we have

Taking into account , we conclude that  is bounded. For a subsequence we can assume that in *X.* Then , that is

From () and (), using again Holder’s inequality, it follows that

and

Therefore, one has

.

From Lemma [2.6](#page6) (iii), is of type , then strongly.

**Proof of Theorem** [**3.2**](#page8). The proof is divided into two steps:

**Step 1**: From Lemmas [3.3](#page8) and [3.4,](#page8) by mountain pass theorem due to Ambrosetti and Rabinowitz [[2],](#page12) there exists a sequence such that

, as . (29)

By Lemma 3.6, that there exists such that is a nontrivial weak solution of problem (1).

**Step 2:** For each , set

so  is nonincreasing. Thus, for any we have that is,

where by From , there exists such that

for all and . By condition , for all and , there exists such that

.

Hence, for all  and , we have



Using the equality it follows that



for all and all . Therefore, we deduce that

for all  and .

From the fact that and , we can choose a function such that

Then, arising as (27) we obtain

for small enough since and . Thus, we obtain

,

where is given by Lemma 3.3 and denote the ball centered at the origin and of radius .

Now, let us choose such that

(30)

Applying Ekeland’s variational principle to the functional , if follows that there exists uch that

By (30) and the fact that

it follows that . From these facts, we have that is a local minimum of the functional defined from onto . Therefore, for and sufficiently small , we have

Letting it following that

we infer that

(32)

From relations (31) and (32), there exists a sequence such that

(33)

In view of Lemma 3.6, is a bounded sequence in . Thus, there exists such that, up to a subsequence, converges strongly to and i.e., is also a nontrivial weak solution for problem (1) such that . The proof of Theorem 3.2 is now complete.

**REFERENCES**

[1] G.A. Afrouzi, V. Radulescu and S. Shokooh (2017), Multiple solutions of Neumann problems: An Orlicz-Sobolev space setting, Bull. Malays. Math. Sci. Soc. 40, 1591-1611.

[2] A. Ambrosetti and P.H. Rabinowitz (1973), Dual variational methods in critical point theory and ap-plications, J. Funct. Anal. 14, 349-381.

[3] G. Autuori, P. Pucci and M.C. Salvatori (2009), Asymptotic stability for anisotropic Kirchhoff systems, J. Math. Anal. Appl. 352, 149-165.

[4] G. Bonanno, G. Molica Bisci, V. Radulescu (2012), Arbitrarily small weak solutions for a nonlinear eigenvalue problem in Orlicz-Sobolev spaces, Monatsh Math. 65, 305-318.

[5] N.T. Chung, H. Q. Toan (2013), On a nonlinear and non-homogeneous problem without (A-R) type condition in Orlicz-Sobolev spaces, Appl. Math. Comput. 219, 7820-7829.

[6] N.T. Chung (2013), Three solutions for a class of nonlocal problems in Orlicz-Sobolev spaces, J. Korean Math. Soc. 50(6), 1257-1269.

[7] F.J.S.A. Correa, G.M. Figueiredo (2009), On a p-Kirchhoff equation via Krasnoselskii’s genus, Appl. Math. Letters 22, 819-822.

[8] G. Dai (2011), Three solutions for a nonlocal Dirichlet boundary value problem involving the p(x)-Laplacian, App. Anal. 92, 1-20.

[9] I. Ekeland (1974), On the variational principle, J. Math. Anal. Appl., 47, 324-353.

[10] X. Fan and D. Zhao (2001), On the spaces  and , J. Math. Anal. Appl. 263, 424-446.

[11] M.G. Huidobro, V.K. Le, R. Manasevich and K. Schmitt (1999), On principal eigenvalues for quasilinear elliptic differential operators: an Orlicz-Sobolev space setting, Nonlinear Diff. Equ. Appl. (NoDEA) 6, 207-225.

[12] G.M. Figueiredo and J.A. Santos (2015), On a nonlocal multivalued problem in an Orlic-Sobolev space via Krasnoselskii’s genus, J. Convex Anal. 22, 447-446.

[13] E.M. Hssini, M. Massar, M. Talbi and N. Tsouli (2013), Infinitely many solutions for nonlocal elliptic p-Kirchhoff type equation under Neumann boundary condition, Int. Journal of Math. Analysis, 7(21), 1011-1022.

[14] E.M. Hssini, N. Tsouli and M. Haddaoui (2017), Existence results for a Kirchhoff type equation in Orlicz-Sobolev spaces, Adv. Pure Appl. Math, 8, 1-12.

[15] A. Kristaly, M. Mihailescu and V. Radulescu (2009), Two non-trivial solutions for a non-homogeneous Neumann problem: an Orlicz-Sobolev space setting. Proc. R. Soc. Edinb. Sect. A 139, 367-379.

[16] G. Kirchhoff , Mechanik, Teubner, Leipzig, Germany, 1883.

[17] V.K. Le (2000), A global bifurcation result for quasilinear elliptic equations in Orlicz-Sobolev spaces, Topol. Methods Nonlinear Anal. 15(2), 301-327.

[18] M. Massar, M. Talbi and N. Tsouli (2014), Multiple solutions for nonlocal system of (p(x); q(x))-Kirchhoff type, Appl. Math. Comput., 242, 216-226.

[19] M. Mihailescu and V. Radulescu (2008), Neumann problems associated to nonhomogeneous differential operators in Orlicz-Sobolev space. Ann. Inst. Fourier Grenoble, 6, 2087-2111.

[20] J. Musielak (1983), Orlicz Spaces and Modular Spaces. Lecture Notes in Mathematics, vol. 1034, Springer, Berlin.

[21] B. Ricceri (2009), A further three critical points theorem, Nonlinear Anal. 71(9), 4151-4157.

[22] M. Struwe (1996), Variational methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian systems, Springer-Verlag, Berlin.

[23] L. Yang (2012), Multiplicity of Solutions for Perturbed Nonhomogeneous Neumann Problem through Orlicz-Sobolev Spaces, Abstract and Applied Analysis Volume 2012, Article ID 236712, 10 pages.

[24] E. Zeidler (1985), Nonlinear Functional Analysis and its Applications, vol. II/B, Berlin, Heidelberg, New York.

**SỰ TỒN TẠI NGHIỆM CHO MỘT LỚP PHƯƠNG TRÌNH KHÔNG THUẦN NHẤT VÀ KHÔNG ĐỊA PHƯƠNG TRONG KHÔNG GIAN ORLICZ-SOBOLEV**

***Tóm tắt****.* *Trong bài báo này, chúng tôi nghiên cứu sự tồn tại đa nghiệm cho một lớp bài toán không thuần nhất và không địa phương trong không gian Orlicz-Sobolev. Các kết quả của chúng tôi ở đây được thiết lập bằng cách dùng định lí qua núi kết hợp với nguyên lí biến phân Ekeland.*

***Từ khóa****: Toán tử không thuần nhất; Không gian Orlicz-Sobolev; Bài toán kiểu Kirchhoff type; Phương pháp biến phân.*

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